We adapt the definitions given in Wilson's book [1].

Definition 1 *A connected graph G is Eulerian if there exists a closed trail containing every edge of G. Such a trail is an* ***Eulerian trail****.*

Definition 2 *A* ***planar graph*** *is a graph that can be drawn in the plane without crossings, that is, so that no two edges intersect geometrically except at a vertex to which both are incident. Any such drawing is a* ***plane drawing****.*

Definition 3 *A planar graph is k-colourable(f) if its faces can be coloured with k colours so that no two faces with a boundary edge in common have the same colour.*

Definition 4 *A connected graph which is both planar and Eulerian is called an* ***H-graph****. (To honour Hakan Ünlü).*

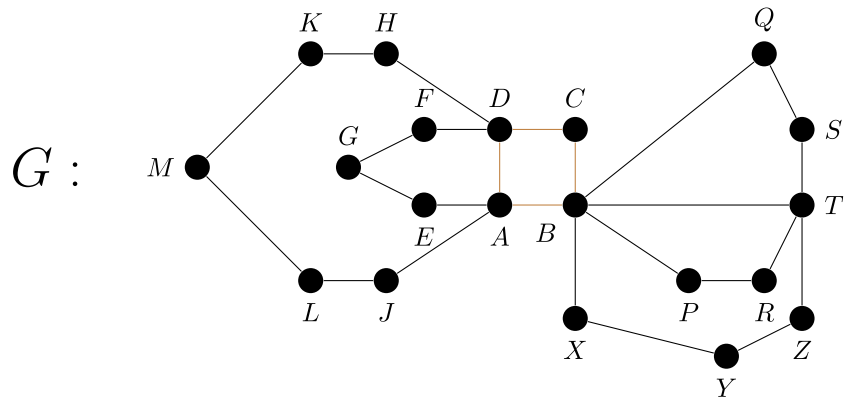
Theorem 5 *An H-graph is 2-colourable(f).*

Proof. The proof is by induction on the the number of faces of the graph.

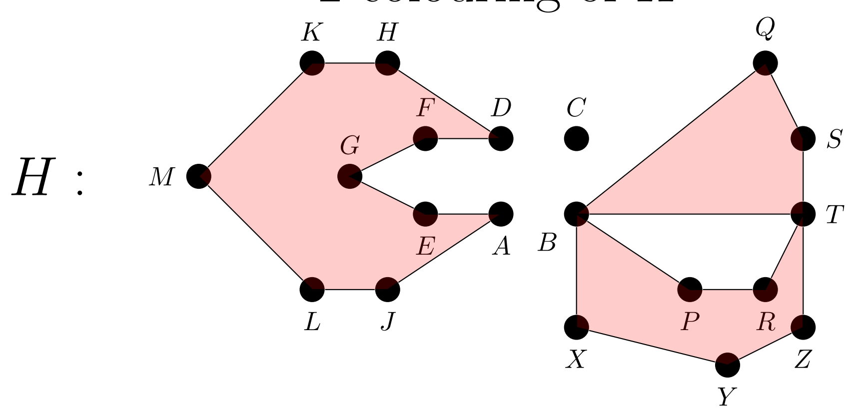
If the graph has only one face, i.e. the unbounded one, then only one colour suffices to color it. So it is 2-colourable(f).

Let G be an H-graph withmore than one face. Clearly one of these faces, say F, is bounded by a cycle C and interior of F contains no other face of G. Let H be the graph obtained from G by removing the edges of C form G. Each connected component of H is clearly an H-graph and each has less faces than G. So, by induction hypothesis, each component can be coloured, say by colours red and white. By recolouring if necessary we may assume that the unbounded face of each component of H is coloured white. Now reinstating C back to obtain G we colour interior of C red and leave every other color used in H unchanged. This will be a 2-coloring of G. This proves the theorem. (See the figure.) ■

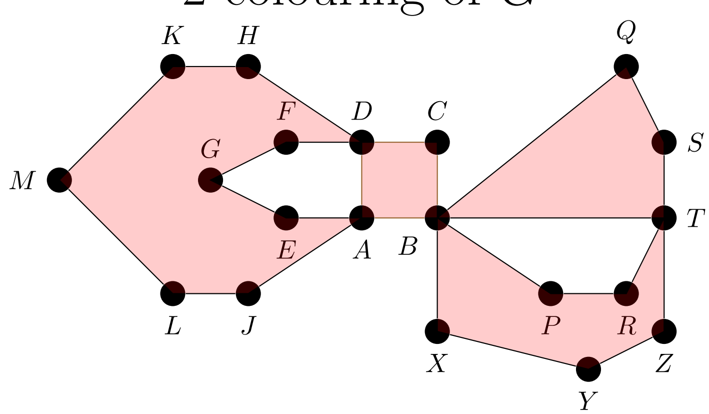
The cycle $\mathcal{C} : A - B - C - D - A$



2-colouring of H



2-colouring of G



References

[1] Robin J. Wilson, Introduction to Graph Theory, Longman,Fourth edition, 1996.